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ON MULTIPLY TRANSITIVE GROUPS

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1. We treat a classification of 4-fold transitive groups. Let G be a 4-fold transitive group on $\Omega=\{1,2,\ldots,n\}$, and set $H=G_{1\,2\,3\,4}$.

The first step of the classification:

Jordan proved that if $H=1$ then $G=S_4$, S_5, A_6 or M_{11}. By this theorem we have that $|I(H)|=4$, 5, 6 or 11 and $N_G(H)^{I(H)}=S_4$, S_5, A_6 or M_{11} respectively. Except the first case the classification is completed.

Theorem 1. [2]

If $N_G(H)^{I(H)}=S_5$, A_6 or M_{11}, then $G=S_5$, A_6 or M_{11} respectively.

The second step of the classification:

Let P be a Sylow 2-subgroup of H. The Jordan's theorem was extended by M.Hall in the following way: If H is of odd order then $G=S_4$, S_5, A_6, A_7 or M_{11}. By this theorem we have that $|I(P)|=4$, 5, 6, 7 or 11 and $N_G(P)^{I(P)}=S_4$, S_5, A_6, A_7 or M_{11} respectively.

Here we give a classification of the special cases in which $|I(P)|=6$ or 11 or $|I(P)|=4$, 5 or 7 and P satisfies some assumptions.

Definition and Notation. Let G be a permutation group on Ω. The stabilizer of points i, j, ..., k in G is denoted by $G_{i\,j\ldots k}$. If X is a subset of G fixing a subset Δ of Ω, then X induces a set of permutation on Δ, which we denote by X^{Δ}. For a subset X of G, I(X) denoteds the set of all the fixed points of X. A G-orbit of minimal length ($\neq 1$) is called a minimal G-orbit.

2. Let G be a 4-fold transitive group and assume that a Sylow 2-subgroup P of $G_{1\,2\,3\,4}$ is not the identity. For a point t of a minimal

P-orbit set $N_G(P_t)^{I(P_t)}=N$. Then N is a permutation group on $I(P_t)$ and satisfies the following conditions:

For any four points i, j, k and l of $I(P_t)$ let R be a Sylow 2-subgroup of $N_{i\,j\,k\,l}$. Then

(1) R is nonidentity semi-regular,

(2) $I(R)=I(P)$.

First we determine the structure of the group N.

Theorem 2. [6,7,8]

Let G be a permutation group on $\Omega=\{1,2,\dots,n\}$ where $n>4$. Assume that a Sylow 2-subgroup P of the stabilizer of any four points in G satisfies the following two conditions:

(i) P is a nonidentity semi-regular group.

(ii) P ifxes exactly r points.

Then

(I) If $r=4$, then $|\Omega|=6$, 8 or 12 and $G=S_6$, A_8 or M_{12} respectively.

(II) If $r=5$, then $|\Omega|=7$, 9 or 13. In particular, if $|\Omega|=9$, then $G\leqq A_9$, and if $|\Omega|=13$, then $G=S_1\times M_{12}$.

(III) If $r=7$ and $N_G(P)^{I(P)}\leqq A_7$, then $G=M_{23}$.

(IV) It is impossible that $r=6$ and $N_G(P)^{I(P)}\leqq A_6$ or $r=11$ and $N_G(P)^{I(P)}\leqq M_{11}$.

By Theorem 2 we have the following

Theorem 3. [6,7,8]

Let G be a 4-fold transitive group on Ω and P be a Sylow 2-subgroup of $G_{1\,2\,3\,4}$.

(I) If $|I(P)|=6$ or 11 then $G=A_6$ or M_{11} respectively.

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(II) Assume that P is not the identity, and for a point t of $\Omega - I(P)$ a Sylow 2-subgroup R of the stabilizer of any four points in $N_G(P_t)^{I(P_t)}$ satisfies the following two conditions:

(i) R is a nonidentity semi-regular group.

(ii) $|I(R)| = |I(P)|$

Then one of the conclutions (I), (II) and (III) in Theorem 2 holds for $N_G(P_t)^{I(P_t)}$. In particular, if t is a point of a minimal P-orbit, then $N_G(P_t)^{I(P_t)}$ satisfies the conditions (i) and (ii).

To prove Theorem 2 we need the following

Theorem 4. [5]

Let G be a 4-fold transitive group on $\Omega = \{1,2,\dots,n\}$. If a Sylow 2-subgroup of $G_{1\,2\,3\,4}$ is semi-regular and not identity, then $G = S_6$, S_7, A_8, A_9, M_{12} or M_{23}.

In the proofs of these theorems we use frequently the conbinatrial argument. For instance the case (II) of Theorem 2 will be proved in the following way.

Assume $|\Omega| > 9$. Let a be an involution of P and $I(P) = \{1,2,3,4,5\}$. We may assume a is of the form

$$a = (1)(2)\dots(5)(6\ 7)(8\ 9)(10\ 11)\dots\ .$$

Since $a \in N_G(G_{6\,7\,8\,9})$, there is an involution b of $G_{6\,7\,8\,9}$ commuting with a. Since $|I(b)| = 5$, we may assume

$$b = (1)(2\ 3)(4\ 5)(6)(7)(8)(9)\dots\ .$$

Since $\langle a,b\rangle < N_G(G_{2\,3\,6\,7})$, there is an involution c of $G_{2\,3\,6\,7}$ commuting with a and b, c is of the form

$$c = (1)(2)(3)(4\ 5)(6)(7)(8\ 9)\dots\ .$$

Then $I(ac)=\{1,2,3,8,9\}$. Hence $\langle a,c\rangle$ is semi-regular on $\{10,11,\ldots,n\}$, and so we may assume

$$a=(1)(2)\ldots(5)(6\ 7)(8\ 9)(10\ 11)(12\ 13)\ldots\ ,$$

$$c=(1)(2)(3)(4\ 5)(6)(7)(8\ 9)(10\ 12)(11\ 13)\ldots\ .$$

Since $\langle a,c\rangle < N_G(G_{10\ 11\ 12\ 13})$, there is an involution d of $G_{10\ 11\ 12\ 13}$ commuting with a and c. We may assume

$$d=(1)(2\ 3)(4\ 5)(6\ 7)(8\ 9)(10)(11)(12)(13)\ldots\ .$$

Since $\langle a.d\rangle < N_G(G_{2\ 3\ 10\ 11})$, there is an involution f of $G_{2\ 3\ 10\ 11}$ commuting with a and d. f is one of the following forms:

(i) $f=(1)(2)(3)(4\ 5)(6\ 7)(8\ 9)(10)(11)(12\ 13)\ldots$,

(ii) $f=(1)(2)(3)(4\ 5)(6\ 8)(7\ 9)(10)(11)(12\ 13)\ldots$.

If f is of the form (i), then

$$af=(1)(2)(3)(4\ 5)(6)(7)(8)(9)\ldots\ .$$

Thus $|I(af)|>5$, which contradicts the assumption. Hence f is of the form (ii). Then

$$cf=(1)(2)(3)(4)(5)(6\ 8\ 7\ 9)\ldots\ .$$

Thus 6, 7, 8 and 9 are contained in the same $G_{I(a)}$-orbit. Since we took 2-cycles (6 7) and (8 9) as arbitarary 2-cycles of a, $G_{I(a)}$ is transitive on $\Omega-I(a)$. Hence for any involution x fixing five points $G_{I(x)}$ is also transitive on $\Omega-I(x)$.

By using this result repeatedly, we can prove that for some point i G_i is 4-fold transitive on $\Omega-\{i\}$. Hence by Theorem 4 $G=S_1\times M_{12}$.

For $|\Omega|\leqq 9$ the proof is similar.

3. By Theorem 3 if G is 4-fold transitive on $\Omega=\{1,2,\ldots,n\}$ and a Sylow 2-subgroup P of $G_{1\ 2\ 3\ 4}$ is not the identity, then $|I(P)|=4$,

5 or 7. In these cases the classification of G is not completed. If P is abelian or transitive on $\Omega - I(P)$ and normal in $G_{1\ 2\ 3\ 4}$, then G is determined:

Theorem 5. [4,8]

If P is a nonidentity abelian group, then $G = S_6$, S_7, A_8, A_9 or M_{23}.

Theorem 6. [1]

If P is a nonidentity normal subgroup of $G_{1\ 2\ 3\ 4}$ and transitive on $\Omega - I(P)$, then $G = S_6$, A_8, M_{12} or M_{23}.

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